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ON CYCLIC TRIGONAL RIEMANN SURFACES. I

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ABSTRACT.

DEFINITION. Call the Riemann surfaces for the equation $y^3 = P(x)$ *cyclic trigonal*. For one case of genus 4 (2 distinct g_3^1's) and all genera greater than 4, cyclic trigonal Riemann surfaces are characterized by the vanishing properties of the theta function at certain (1/6)-periods of the Jacobian. Also for trigonal Riemann surfaces of genera 5, 6, and 7, homogeneous theta relations are derived using the fact that Prym varieties for trigonal Riemann surfaces are Jacobians.

1. Introduction. A closed Riemann surface W which can be realized as a three-sheeted covering of the Riemann sphere $\mathbf{P}_1$ is called *trigonal*. If there is an automorphism of period three on W which permutes the sheets of the covering, we shall call W *cyclic trigonal*. This latter case is equivalent to the condition that all the branch points of the three-sheeted covering have multiplicity three and W is the Riemann surface for the equation

$$y^3 = \prod_{i=1}^{s}(x-\alpha_i)\prod_{j=1}^{t}(x-\beta_j)^2 \tag{1.1}$$

where the $s+t$ complex numbers α_i, β_j are distinct and finite. We assume $s + 2t \equiv 0 \pmod 3$ so there is no branching over ∞ in $\mathbf{P}_1$. The genus of W is $p = s + t - 2$.

In this paper we derive necessary and sufficient conditions for a Riemann surface of genus p to be cyclic trigonal for $p \geqslant 5$ and for one case of genus 4. The case $p = 2$ is done in [**2**, p. 94]. The cases $p = 3$ and the second case of $p = 4$ are open.

The characterization will be in terms of vanishings of the associated theta function at (1/6)-periods of the Jacobian of W, $J(W)$. This will extend to trigonal cyclic Riemann surfaces an analysis analogous to the classical theory of hyperelliptic theta functions when hyperellipticity is characterized by certain vanishings of the theta function at half periods of $J(W)$ [**11**].

By Torelli's theorem the conformal type of a Riemann surface is determined by the symplectic equivalence class of its period matrices. These results for hyperelliptic and cyclic trigonal Riemann surfaces can be viewed as part of a program of extracting from a period matric (in this case via the theta function) information about special properties a Riemann surface might possess.

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Part II of this paper will be devoted to deriving the equation ((1.1)) of the cyclic trigonal Riemann surface from functions of theta constants, again extending to these surfaces an analysis similar to the hyperelliptic case [**8**, p. 326; **16**].

In §2 we introduce notation, mention some known results, and prove some preliminary lemmas.

In §3 we derive the high order vanishing properties of the cyclic trigonal theta function.

In §4 we show that these vanishing properties characterize the existence of a trigonal cyclic covering for $p \geqslant 5$. This analysis appears to be of necessity quite a bit more involved than that of the hyperelliptic case. One must first show the existence of a g_3^1 and then show that the three-sheeted covering of $\mathbf{P}_1$ is cyclic. Just as one must divide the hyperelliptic analysis into cases of even and odd genus, so one divides the present analysis according to the residue modulo three of the genus. Each of the three cases presents slightly different problems, and in each case the lower genera present additional problems. Hence the length of this paper.

In §5 we deal with the one case of $p = 4$ that we are able to handle.

Finally, in §6 we include some additional remarks about trigonal Riemann surfaces of genus 5, 6, and 7. Here the idea is to define the trigonal locus in Teichmüller space by equations in thetanulls. This section can be read independently of the rest of the paper.

2. Definitions, notation, and preliminary results. In this section we assume the reader is already familiar with the subject of Riemann surfaces. Mainly we will consider those points where notation, definitions, and results are not standard.

W_p will stand for a closed Riemann surface of genus p. Let $A(W_p)$ denote the full group of automorphisms (conformal self-maps) of W_p. If G is a finite group of automorphisms, the space of orbits, denoted W_p/G, is naturally a Riemann surface. The map $W_p \to W_p/G$, which takes a point of W_p into its G-orbit, is an n-sheeted covering of closed Riemann surfaces, where n is the order of G. Such a covering will be called *normal.*

An arbitrary n-sheeted covering $W_p \to W_q$ will be called *strongly branched* if $p > n^2q + (n-1)^2$ [**1**]. In this case, if n is prime, there is only one such covering for given n and q.

If $W_p \to W_q$ is n-sheeted and $W_p \to W_{q'}$ is n'-sheeted then the inequality of Castelnuovo and Severi says that $p \leqslant nq + n'q' + (n-1)(n'-1)$ under suitable conditions. For our purposes we note that the inequality holds if $(n, n') = 1$ or if $n = n'$ is a prime number. Some consequences are stated in the following lemma.

LEMMA 2.1. *Suppose $W_p \to W_q$ is an n-sheeted covering.*

(i) *If n is prime and $p > 2nq + (n-1)^2$ then there is only one such cover with the given n and q.*

(ii) *If $W_p \to W_{q'}$ is an n'-sheeted covering where $q' = 0$, $n' = 3$ and $n = 2$ then $p \leqslant 2q + 2$.*

(iii) *If $n = 2$ and $p > 4q + 1$ then there is a unique automorphism T of period two so that the genus of $W/\langle T \rangle$ is q.*

REMARK. In the light of Lemma 2.1(i) a better definition of strongly branched might be $p < 2nq + (n - 1)^2$. But with this definition all the results of [1] do not appear to follow.

If $W_p \to W_q$ is an n-sheeted cover where n is prime and $p > 2nq + (n - 1)^2$, let N be the normal subgroup of $A(W_p)$ which leaves the fibers of the cover fixed. Then $A(W_p)/N$ is isomorphic to a finite subgroup of $A(W_q)$. If $G \subset A(W_p)$ and $G \cap N = \langle e \rangle$, then we say that G *descends* to W_q. That means that G permutes the fibers of the map $W_p \to W_q$ and induces a group of automorphisms on W_q isomorphic to G.

A divisor D on W_p is a formal finite sum $\Sigma n_i z_i$ where $z_i \in W_p$ and $n_i \in Z$. The degree of D, deg D, is Σn_i. D is called integral if $n_i \geqslant 0$ for all i. If ϕ: $W_p \to W_q$ is an n-sheeted cover and $D = \Sigma n_i z_i$ is a divisor on W_q, we call $\Sigma n_i \phi^{-1}(z_i)$ the *lift* of D. The branch points in $\phi^{-1}(z_i)$ are counted according to multiplicity so deg $\phi^{-1}(z_i) = n$ always. If $D = \Sigma n_i z_i$ is a divisor on W_p we call $\Sigma n_i \phi(z_i)$ the *drop* of D to W_q. Thus if D is a divisor on W_q the drop of the lift of D is nD.

If D is (linearly) equivalent to an integral divisor then $|D|$ will denote the complete linear series of integral divisors equivalent to D, denoted g_n^r, where $n = \deg D$ and r is the projective dimension. (We will frequently abuse notation by writing $D \equiv g_n^r$ for $|D| = g_n^r$.) We will denote the canonical series $g_{(2p-2)}^{p-1}$ by K when convenient. If $g_n^r + g_{n'}^{r'} \equiv K$ we say one of these linear series is the *complement* of the other. For complete linear series g_n^r the Riemann-Roch theorem says $r = n - p + i$, where i is the index of speciality. If g_n^r is special ($i > 0$) then Clifford's theorem says that $n - 2r \geqslant 0$.

Linear series may be complete or incomplete, may or may not have fixed points, and may be simple or composite. We shall discuss the latter two concepts a little more fully. A g_n^r, $r \geqslant 2$, will be called simple if the map of W_p into $\mathbf{P}_r$ (projective space of dimension r) is one-to-one in general. If g_n^r is not simple it is called composite. This means that there is a cover ϕ: $W_p \to W_q$ of t sheets, $t \geqslant 2$, and a simple linear series $g_{(n-\varepsilon)/t}^r$ on W_q, the divisors of which lift to the nonfixed points of g_n^r. ε is the number of fixed points of g_n^r. If g_n^r is complete, so is $g_{(n-\varepsilon)/t}^r$. The set of fibers of the map ϕ will be called an *involution* and denoted γ_t^1. We say that g_n^r is *compounded* of the involution γ_t^1. If $q = 0$ then $\gamma_t^1 = g_t^1$.

A linear series g_{p-1}^r such that $2g_{p-1}^r \equiv K$ will be called *half-canonical* and be abbreviated $(1/2)$-K. If $2ng_{p-1}^r \equiv nK$ we say g_n^r is $(1/2n)$-nK. We will be particularly concerned with $(1/6)$-$3K$ linear series which includes the possibility of $(1/2)$-K.

LEMMA 2.2 (CASTELNUOVO'S INEQUALITY AND THEOREM). *Suppose W_p admits a simple g_n^r. Then*

$$p \leqslant \frac{n - r + \varepsilon}{r - 1} \frac{n - 1 - \varepsilon}{2}, \tag{2.1}$$

where $\varepsilon = 0, 1, 2, \ldots,$ or $r - 2$ and $n - r + \varepsilon \equiv 0 \pmod{r - 1}$. If we have equality in (2.1) *then W_p admits a g_m^1 which imposes two linear conditions on g_n^r* [5].

The main use we will make of the second part of Lemma 2.2 is the following.

LEMMA 2.3. *If W_{3r+1} admits a simple g^r_{3r} which is not $(1/2)$-K, then $2g^r_{3r} \equiv g^{3r-1}_{6r} \not\equiv K$ and we have equality in* (2.1). *Then W_{3r+1} admits a g^1_3 or a g^1_4. If W_{3r+1} admits a g^1_3 and $g^r_{3r} + h^r_{3r} \equiv K$, then one of these two linear series is rg^1_3.*

A Riemann surface admitting a g^1_3 without fixed points will be called *trigonal.* A Riemann surface which is a two-sheeted covering of a surface of genus q will be called *q-hyperelliptic* (abbreviated (q-H)). Thus, by Lemma 2.1(ii), if W_p is (q-H) and trigonal then $p \leqslant 2q + 2$. Thus a Riemann surface of genus five cannot be (1-H) (elliptic-hyperellitpic) and trigonal simultaneously.

By the Riemann-Roch theorem it follows that a canonical divisor containing $n - 1$ points of a divisor of a g^1_n must contain the nth. Applying this for $n = 3$ yields the following lemma.

LEMMA 2.4 [**4**]. *Suppose g^r_n and $g^{r'}_{n'}$ are complementary linear series on a trigonal Riemann surface. Then one of these is compounded of the g^1_3. A $(1/2)$-K linear series on a trigonal Riemann surface is compounded of the g^1_3.*

If all the branch points of the covering $W_p \to \mathbf{P}_1$ given by a g^1_3 are of multiplicity three then the covering is normal with group Z_3. In this case W_p will be said to be *cyclic trigonal.* (This is one of the few cases where the multiplicity of the branching alone suffices to insure a normal covering.) We will call a branch point of multiplicity three a 3-*branch point.* Thus a trigonal Riemann surface of genus p with $p + 2$ 3-branch points in the three-sheeted cover of $\mathbf{P}_1$ will be cyclic trigonal by the Riemann-Hurwitz formula.

If $p \geqslant 5$ W_p admits at most one g^1_3 without fixed points and a W_4 admits at most two which are complementary.

Let $J(W_p)$ be the Jacobian of W_p [**12**]. If we fix a basepoint in W_p then we have the usual map of W_p and of divisors on W_p into $J(W_p)$. If we fix a canonical homology basis on W_p, then we have determined a period matrix $(\pi i E, B)^{p \times 2p}$ and we can form the theta function $\theta(u; B)$, where $u \in C^p$ and B can be considered a function on, say, Teichmüller space.

We often make a statement that "the theta function" has such and such a property. When we do, we assume a basepoint chosen, if necessary, and a canonical homology basis chosen. Then there is a uniquely defined theta function (with zero theta characteristic). The stated property will, however, be independent of the choices. For instance, for genus three, the following is such a statement: W_3 is hyperelliptic if and only if the theta function vanishes at a half period to order two.

If the theta function vanishes at a point $F \in J(W_p)$ to order k, we will often say that $\theta(F) = 0$ *k times*. This is well defined since the theta divisor is well defined on $J(W_p)$.

Basic to all this analysis is Riemann's theorem [**13**], parts of which are collected in the following theorem.

THEOREM 2.5. *Suppose a basepoint and a canonical homology basis have been chosen for W_p and the map u: $W_p \to J(W_p)$ is defined. Then there is a point κ in $J(W_p)$ so that:*

(a) *For any $F \in J(W)$ there is a divisor D of degree p on W_p so that*

$$u(D) + \kappa \equiv F. \tag{2.2}$$

(b) $i(D) = 0$ *if and only if* $\theta(F) \neq 0$, *in which case D is the unique solution to* (2.2).

(c) *If* $\theta(F) = 0$ $r+1$ *times then* (2.2) *can be solved by divisors D such that* $|D| = g^r_{p-1}$. *Such a* g^r_{p-1} *will be said to correspond to F.*

(d) *If* $2nF \equiv 0$ *in part* (c) *then* g^r_{p-1} *is* $(1/2n)$-nK.

(e) *If* $\theta(F) = \theta(-F) = 0$, $u(D) + \kappa \equiv F$ *and* $u(D') + \kappa \equiv -F$, *then* $D + D' \equiv K$.

(f) $u(K) + 2\kappa \equiv 0$.

Now we apply Riemann's theorem to smooth abelian covers [**2**]. Suppose ϕ: $W_p \to W_q$ is an n-sheeted unbranched normal covering where the Galois group G is abelian. Let a denote the map that lifts divisors of degree zero from W_q to W_p. Suppose basepoints z, z_0 chosen so that $\phi(z) = z_0$. Then there is an induced map, also denoted a, a: $J(W_q) \to J(W_p)$ so that the following diagram commutes [**2**, p. 4] ($\mathfrak{D}_0$ denotes divisors of degree zero):

$$\begin{array}{ccc} \mathfrak{D}_0(W_p) & \xrightarrow{u} & J(W_p) \\ \uparrow a & & \uparrow a \\ \mathfrak{D}_0(W_q) & \xrightarrow{u_0} & J(W_q) \end{array}$$

Then $\ker a$ ($\subset J(W_q)$) is isomorphic to G. Moreover, to every finite subgroup A of $J(W_q)$ there is a suitable smooth abelian cover of W_q so that the diagram commutes. In this case we will say that the cover $W_p \to W_q$ *corresponds* to the finite subgroup A.

We need to know more about this situation.

LEMMA 2.6. *In the immediately preceding situation, suppose there is a point* $E \in J(W_q)$ *so that the theta function vanishes at the points of the coset* $E + \ker a$ *to orders* $r_1 + 1, r_2 + 1, \ldots, r_n + 1$. *Then the theta function for* $J(W_p)$ *vanishes at aE to order* $n + \Sigma r_i$. *Moreover, if* $r_i \geqslant 0$ *for* $i = 1, \ldots, n$ *and one of the linear series* $g^{r_i}_{q-1}$ *on* W_q *corresponding to a period in* $E + \ker a$ *is simple, then the* $g^{\Sigma r_i + n - 1}_{p-1}$ *on* W_p *corresponding to aE in* $J(W_p)$ *is also simple.*

PROOF [**2**, p. 74]. The last statement is not proven in the reference. It follows from familiar procedures.

We need to quote another theorem from [**2**, p. 77].

LEMMA 2.7. *Suppose* $W_p \in$ (q-H). *This property is characterized by the vanishing properties of the theta function at certain half-periods if*:
when p is odd $6q \leqslant p + 3$,
when p is even $6q \leqslant p$.

If W_p is trigonal ($p \geqslant 5$) and also (q-H) then we know $p \leqslant 2q + 2$ (Lemma 2.1(ii)). Consequently p and q cannot be related as in Lemma 2.7. Thus W_p cannot have the vanishing properties which would insure that $W_p \in$ (q-H) for q in the ranges of Lemma 2.7. We shall refer to this as the *nonvanishing properties* of a trigonal Riemann surface.

We conclude this section with some more results on a W_{3r+1} admitting a g^r_{3r} (in addition to Lemma 2.3). The following lemma will be used to exclude the possibility of a g^1_4 in applying Lemma 2.3 later in the paper.

LEMMA 2.8. *Suppose $p = 3r + 1$ and W_p admits a fixed point free automorphism T of period three. Then W_p cannot admit a unique g_4^1.*

PROOF. Suppose W_p admits a unique g_4^1. Then T permutes the divisors of g_4^1. Since g_4^1 is parametrized by $\mathbf{P}_1$, there is at least one divisor D_0 in g_4^1 so that $TD_0 = D_0$. Thus T permutes the points of D_0, preserving multiplicities. We conclude that T has a fixed point among the points of D_0 since $(3, 4) = 1$, contradiction. Q.E.D.

LEMMA 2.9. *Suppose W_{3r+1} admits a simple $(1/2)$-K g_{3r}^r, where $r \geqslant 10$. Thus W_{3r+1} admits a g_3^1, a g_4^1, or a γ_3^1 which is the set of fibers of a three-sheeted cover of a torus W_1* [7]. *In the last case $g_{3r}^r = | g_{3r}^{r-1} |$, where g_{3r}^{r-1} is the lift of a g_r^{r-1} on W_1.*

REMARK. With the stated hypotheses (which are more restrictive than those of [7]), a g_3^1 is excluded by Lemma 2.4.

PROOF. For the first part of the proof we cite the reference. For the last statement we refer to a generalization of the Riemann-Roch theorem due to Castelnuovo [**6**, p. 324] to infer that any divisor D_1 in γ_3^1 imposes two conditions on g_{3r}^r. But by [**3**] two divisors in γ_3^1, $D_1 + D_2$, impose at most three conditions on g_{3r}^r since $| D_1 + D_2 | = g_6^1$ on W_{3r+1}. Thus any D_2 in γ_3^1 imposes one condition on $g_{3r}^r - D_1 = g_{3r-3}^{r-2}$. That is, g_{3r-3}^{r-2} is composite, being compounded of γ_3^1. Since $D_1 + g_{3r-3}^{r-2}$ is the lift of a g_r^{r-1} from W_1, the result follows. Q.E.D.

LEMMA 2.10. *Suppose W_{3r+1} admits a simple $(1/2)$-K g_{3r}^r and the γ_3^1 of Lemma 2.9. Then there are three additional complete $(1/2)$-K linear series of dimension $r - 1$.*

PROOF. There is a g_r^{r-1} on W_1 whose lift when completed is g_{3r}^r. On W_1 there are three other linear series, h_r^{r-1}, k_r^{r-1}, l_r^{r-1}, so that their doubles are linearly equivalent to $2g_r^{r-1}$. When lifted, the three linear series become half-canonical since the lift of g_r^{r-1} is. If the lift of h_r^{r-1} is denoted h_{3r}^{r-1}, we see it is complete since g_{3r}^r is the unique linear series of that dimension and degree. Q.E.D.

REMARK. It also follows that the sum of the four $(1/2)$-K linear series on W_p is bicanonical.

LEMMA 2.11. *Suppose W_{3r+1}, $r \geqslant 2$, admits a g_3^1 without fixed points, and rg_3^1 is the unique $(1/2)$-K g_{3r}^r. Then there does not exist a complete $(1/2)$-K h_{3r}^{r-1}.*

PROOF. Suppose there does exist a complete $(1/2)$-K h_{3r}^{r-1}. Then $h_{3r}^{r-1} \equiv (r - 1)g_3^1 + x + y + z$, where x, y, and z are three points on W_{3r+1} (Lemma 2.4). Since $2h_{3r}^{r-1} \equiv 2g_{3r}^r \equiv 2rg_3^1$, we see that $2x + 2y + 2z \equiv 2g_3^1 \equiv g_6^2$, which is necessarily complete and composite. Then $2x + 2y + 2z = D_1 + D_2$ when D_1 and D_2 are divisors in g_3^1. Thus two of the points, say $x + y$, must be in D_1. It follows that we can assume $D_1 = 2x + y$ and $D_2 = y + 2z$. Therefore $D_1 = D_2$, $x = z$ and $x + y + z = 2x + y = D_1$. Thus the dimension of h_{3r}^{r-1} must be r, a contradiction. Q.E.D.

This property proven in Lemma 2.11 will also be referred to as a nonvanishing property for the theta function for such a trigonal Riemann surface. For on the Jacobian the theta function does not vanish to order r at any half-period.

3. Vanishing properties of the cyclic trigonal theta functions. Let W_p be the Riemann surface for the equation

$$y^3 = \prod_{i=1}^{s} (x - \alpha_i) \prod_{j=1}^{t} (x - \beta_j)^2 \tag{3.1}$$

where the $s + t$ complex numbers α_i, β_j are distinct and finite. Without loss of generality we assume $s + 2t \equiv 0 \pmod 3$ and $s \leqslant t$. Consequently, the three-sheeted covering is unbranched over ∞ and $p = s + t - 2$. On W_p y is a single valued function whose divisor can be written

$$(y) = \sum_{i=1}^{s} a_i + \sum_{j=1}^{t} 2b_j - \left(\frac{s+2t}{3}\right)(\infty_1 + \infty_2 + \infty_3), \tag{3.2}$$

where a_i (respectively b_j) is the point on W_p over α_i (respectively β_j) and $\infty_1 + \infty_2 + \infty_3$ is the divisor of three distinct points over ∞. (When convenient we shall denote a_i by c_i for $i = 1, 2, \ldots, s$ and b_j by c_{j+s} for $j = 1, 2, \ldots, t$.)

A canonical divisor K on W_p can be written

$$K = 2\sum_{i=1}^{s} a_i + 2\sum_{j=1}^{t} b_j - 2(\infty_1 + \infty_2 + \infty_3). \tag{3.3}$$

Writing g_3^1 for $|\infty_1 + \infty_2 + \infty_3|$ and using the fact that $\Sigma a_i + 2\Sigma b_j \equiv ((s + 2t)/3)g_3^1$, we see that

$$\begin{aligned} K &\equiv 2\Sigma a_i - \Sigma a_i + \left(\left(\frac{s+2t}{3}\right) - 2\right) g_3^1 \\ &\equiv \left(\frac{2t - 2s - 6}{3}\right) g_3^1 + s g_3^1 + \Sigma a_i \\ &\equiv 2\left(\left(\frac{t - s - 3}{3}\right) g_3^1 + 2\Sigma a_i\right) \end{aligned}$$

since $g_3^1 = |3a_i|$ for any i. If $t > s$ let $r_0 = (t - s - 3)/3$ or $t - s = 3(r_0 + 1)$. Thus for $t > s$ the following is half-canonical:

$$g_{p-1}^{r_0} = r_0 g_3^1 + 2\sum_{i=1}^{s} a_i. \tag{3.4}$$

We shall see shortly why it has dimension precisely r_0.

Let z_0 in W_p be a basepoint for the map u: $W_p \to J(W_p)$ and let $|z_0 + z_1 + z_2| = g_3^1$. ($z_0$ may or may not be an a_i or b_j.) Now pick a point G_0 in $J(W_p)$ so that $3G_0 \equiv u(g_3^1)$. The choices of z_0 and G_0 are made one and for all. The results of this paper, however, will not depend on these choices.

Now $|3c_k| = g_3^1$, $k = 1, 2, \ldots, s + t$, so there are points C_k in $J(W_p)$ satisfying the equation

$$u(c_k) = C_k + G_0 \qquad (3C_k \equiv 0). \tag{3.5}$$

(Write $A_i = C_i$ for $i = 1,\ldots,s$ and $B_j = C_{s+j}$ for $j = 1,2,\ldots,t$.) On W_p $K \equiv 2\Sigma c_k - 2g_3^1$, so in $J(W_p)$ we have

$$\begin{aligned} -2\kappa = u(K) &= 2\Sigma u(c_k) - 2u(g_3^1) \\ &= 2\Sigma C_k + 2(s+t-3)G_0 = 2\Sigma C_k + 2(p-1)G_0 \end{aligned}$$

Thus $\kappa = -\Sigma C_k - (p-1)G_0 + E_2$, where $2E_2 \equiv 0$ in $J(W_p)$.

DEFINITION. $E_1 = E_2 + \Sigma_{k=1}^{s+t} C_k$ in $J(W_p)$.

Consequently, $6E_1 \equiv 0$, $E_1 + E_2 = \Sigma C_k = \Sigma A_i + \Sigma B_j$ and $\kappa = -E_1 - (p-1)G_0$. Thus if $\varepsilon_i \geqslant 0$, $\Sigma \varepsilon_i = p - 1$ then

$$u(\Sigma \varepsilon_i c_i) + \kappa = -(E_1 - \Sigma \varepsilon_i C_i). \tag{3.6}$$

LEMMA 3.1. $2E_1 \equiv 0$ *if and only if* $\Sigma_{k=1}^{s+t} C_k \equiv 0$.

This follows directly from the definition. By (3.2) and (3.5) we have

$$\Sigma A_i + 2\Sigma B_j + (s+2t)G_0 - (s+2t)G_0 \equiv 0,$$

so

$$\Sigma A_i \equiv \Sigma B_j, \qquad E_1 + E_2 \equiv 2\Sigma A_i \equiv 2\Sigma B_j,$$

and

$$2E_1 \equiv \Sigma A_i \equiv \Sigma B_j. \tag{3.7}$$

It now follows from (3.4) that if $r_0 \geqslant 0$,

$$u(g_{p-1}^{r_0}) + \kappa \equiv E_2. \tag{3.8}$$

LEMMA 3.2. *Suppose* $\Sigma_{i=1}^{s} \varepsilon_i A_i + \Sigma_{j=1}^{t} \delta_j B_j \equiv 0$, *where*

(i) $\varepsilon_i, \delta_j = 0, 1,$ *or* 2,

(ii) $\Sigma \varepsilon_i + \Sigma \delta_j \equiv 0 \pmod 3$.

Then only one of the following three cases holds.

(i) $\varepsilon_1 = \cdots = \varepsilon_s = \delta_1 = \cdots = \delta_t = 0$,

(ii) $\varepsilon_1 = \cdots = \varepsilon_s = 1$; $\delta_1 = \cdots = \delta_t = 2$,

(iii) $\varepsilon_1 = \cdots = \varepsilon_s = 2$; $\delta_1 = \cdots = \delta_t = 1$.

PROOF. Suppose $\Sigma \varepsilon_i + \Sigma \delta_j = 3\gamma$. Then

$$\Sigma \varepsilon_i (u(a_i) - G_0) + \Sigma \delta_j (u(b_j) - G_0) \equiv 0$$

or

$$\Sigma \varepsilon_i u(a_i) + \Sigma \delta_j u(b_j) \equiv (\Sigma \varepsilon_i + \Sigma \delta_j) G_0 \equiv \gamma u(g_3^1).$$

Therefore, there is a function z on W_p whose divisor is

$$\Sigma \varepsilon_i a_i + \Sigma \delta_j b_j - \gamma(\infty_1 + \infty_2 + \infty_3)$$

and

$$z^3 = \Pi(x - \alpha_i)^{\varepsilon_i} \Pi(x - \beta_j)^{\delta_j}.$$

If $p \geqslant 5$, g_3^1 is unique and the result follows. For general p the result follows by observing that the a_i's and b_j's are fixed points for a unique automorphism of period three since the stabilizer of any point in the full group of automorphisms on a Riemann surface is cyclic. Q.E.D.

Referring to (3.4), an argument analogous to the above shows that no subsum of $2\Sigma a_i$ is linearly equivalent to g_3^1. By Lemma 2.4 if follows that the half-canonical linear series $g_{p-1}^{r_0}$ must be complete.

We now consider (1/6)-periods in $J(W_p)$ of the type

$$(3.9) \qquad E_1 - \sum_{i=1}^{s} \varepsilon_i A_i - \sum_{j=1}^{t} \delta_j B_j,$$

where $\varepsilon_j, \delta_j = 0, 1$, or 2 and $\Sigma\varepsilon_i + \Sigma\delta_j \equiv s + t \pmod 3$.

First we count that there are 3^p such periods ($p = s + t - 2$). For there are 3^{p+1} ways of writing periods in the form (3.9). The relation $\Sigma A_i \equiv \Sigma B_j$, and the fact that there are essentially no other relations of this type (Lemma 3.2), means that there are actually 3^p distinct periods of type (3.9). Notice that three times any period of type (3.9) is the half-period E_2. Also $3E_1 \equiv E_2$. If we subtract $E_2 = E_1 - \Sigma_{k=1}^{s+t} C_k$ from all of the 3^p periods of type (3.9) we obtain a group of 3^p (1/3)-periods which we shall call $\mathcal{G}$. Thus the (1/6)-periods of type (3.9) are the coset $E_2 + \mathcal{G}$ in $J(W_p)$, a coset closed under taking negatives.

To derive the vanishing properties of the theta function for W_p we refer to [**2**, Corollary 3, p. 30]. This corollary tells us the vanishing, or possibly nonvanishing, of the theta function at (1/6)-periods of type (3.9). (For an integer n, let $\bar{n}$ be the smallest nonnegative residue of n modulo 3.) Let

(3.10) (i) $\overline{\Sigma\varepsilon_i + \Sigma\delta_j} = s + t - 3\tau_0$,

(3.10) (ii) $\Sigma(\overline{\varepsilon_i + 1}) + \Sigma(\overline{\delta_j + 2}) = s + t - 3\tau_1$,

(3.10)(iii) $\Sigma(\overline{\varepsilon_i + 2}) + \Sigma(\overline{\delta_j + 1}) = s + t - 3\tau_2$.

Then the theta function vanishes at the indicated (1/6)-period to order $\Sigma_{i=0}^{2} \max(0, \tau_i)$. Since $\tau_0 + \tau_1 + \tau_2 = 0$, one sees this is simply the maximum of the three numbers $|\tau_0|, |\tau_1|, |\tau_2|$.

Now we shall rewrite the period (3.9) as

$$E_1 - \sum_{\mathcal{S}_1}^{s_1} A - \sum_{\mathcal{S}_2}^{s_2} 2A - \sum_{\mathcal{T}_1}^{t_1} B - \sum_{\mathcal{T}_2}^{t_2} 2B,$$

where $\mathcal{S}_1$ is the subset of indices in (3.9) where $\varepsilon_i = 1$, card $\mathcal{S}_1 = s_1$; $\mathcal{S}_2$ is the subset of indices in (3.9) where $\varepsilon_i = 2$, card $\mathcal{S}_2 = s_2$; $\mathcal{T}_1$ is the subset of indices in (3.9) where $\delta_j = 1$, card $\mathcal{T}_1 = t_1$; $\mathcal{T}_2$ is the subset of indices in (3.9) where $\delta_j = 2$, card $\mathcal{T}_2 = t_2$. Let $\mathcal{S}_0$ be the subset of indices in (3.9) where $\varepsilon_i = 0$, card $\mathcal{S}_0 = s_0$, and let $\mathcal{T}_0$ be the subset of indices in (3.9) where $\delta_j = 0$, card $\mathcal{T}_0 = t_0$. Then $s_0 + s_1 + s_2 = s$ and $t_0 + t_1 + t_2 = t$. We rewrite (3.10) in the form

(3.11) (i) $s_1 + 2s_2 + t_1 + 2t_2 = s + t - 3\tau_0$,

(3.11) (ii) $s_0 + 2s_1 + t_2 + 2t_0 = s + t - 3\tau_1$,

(3.11)(iii) $s_2 + 2s_0 + t_0 + 2t_1 = s + t - 3\tau_2$,

or

(3.12) (i) $3\tau_0 = (s_0 - t_2) - (s_2 - t_0) = \mu_0 - \mu_2$,

(3.12) (ii) $3\tau_1 = (s_2 - t_0) - (s_1 - t_1) = \mu_2 - \mu_1$,

(3.12)(iii) $3\tau_2 = (s_1 - t_1) - (s_0 - t_2) = \mu_1 - \mu_0$, where

$$\mu_0 = s_0 - t_2, \quad \mu_1 = s_1 - t_1 \quad \text{and} \quad \mu_2 = s_2 - t_0.$$

Thus $3\tau = \max|\mu_i - \mu_j|$ where the theta function vanishes at the (1/6)-period (3.9) to order τ.

From (3.12) we could derive all the vanishing properties of the theta function at periods of type (3.9). For instance, we see that the theta function does not vanish if $\mu_0 = \mu_1 = \mu_2 = (t-s)/3$. Also $\theta(E_2) = 0$ $(t-s)/3$ times. However, we shall limit ourselves to deriving those high order vanishing properties that will characterize the existence of a cyclic trigonal Riemann surface.

Just as one usefully divides the cases for the hyperelliptic vanishing properties into odd and even genera, so we shall divide the genera according to their residues modulo three. Also the case $p \equiv 1 \pmod 3$ is further divided into two cases since the existence of a half-canonical g_{3r}^r $(p = 3r+1)$ introduces further complications. Consequently, we distinguish four cases.

Case 1. $p \equiv 1 \pmod 3$ and $s > 0$. Let $p = 3r+1$. In this case the theta function vanishes at the (1/6)-period E_1 to order $r+1$. E_1 is not a (1/2)-period. At (1/6)-periods of type

$$E_1 - (C_k + C_l + C_m) \quad \text{or} \quad E_1 - (2C_k + C_l), \tag{3.13}$$

the theta function vanishes to order r (except when $s = 3$ and $C_k + C_l + C_m = A_1 + A_2 + A_3$). Since $2E_1 = \Sigma A_i = \Sigma B_j$ by (3.7) we see that none of these (1/6)-periods are (1/2)-periods. There are two aspects of these high order vanishings that we will consider.

The first is the number of (1/6)-periods of type (3.13). This number is clearly

$$\binom{p+2}{3} + 2\binom{p+2}{2}.$$

However, recall that if $\theta(F) = 0$ then $\theta(-F) = 0$ also. It is possible for two of the periods of type (3.14) to sum to zero. For example, if $s = 6$ then

$$(E_1 - (A_1 + A_2 + A_3)) + (E_1 - (A_4 + A_5 + A_6)) = 0$$

by (3.7). What we really want to count is the number of pairs $\{F, -F\}$ when $\theta(F) = 0$ and F is of type (3.13). The following lemma is stated in a form useful for the later characterizations of §4.

LEMMA 3.3. *Let $p = 3r+1$, $r \geq 2$. If W_p is cyclic trigonal with $s > 0$ then there are more than $\binom{p+1}{3} + 2\binom{p+1}{2}$ pairs $\{F, -F\}$ where F is of type* (3.13) *so that the theta function vanishes to order r at F.*

PROOF. We enumerate all cases in tables. We must account for all (1/6)-periods of type (3.13). In the following tables F stands for a (1/6)-period of type (3.13) except that we also include the vanishings to order $r+1$. If $-F$ is also of type (3.13) it is written in. A blank for $-F$ means it is not of type (3.13). The indices on the A's and B's are to be taken as representative of the possibilities. The numbers in the last column are derived by permutations of the indices over the possibilities. For example, in the second row of the first table below, where $s = 3$ and $t = 6$, $E_1 - (A_1 + A_2 + B_1)$ really represents 18 possibilities since there are three choices for the A's and six choices for the B's.

Case. $s = 3$, $t = 6$, $p = 7$.

Order of vanishing of the theta function	F	$-F$	number of pairs $\{F, -F\}$
$r + 1$	$E_1 - (A_1 + A_2 + A_3)$	E_1	1
r	$E_1 - (A_1 + A_2 + B_1)$	$E_1 - (A_3 + 2B_1)$	18
r	$E_1 - (A_1 + B_1 + B_2)$		45
r	$E_1 - (B_1 + B_2 + B_3)$	$E_1 - (B_4 + B_5 + B_6)$	10
r	$E_1 - (2A_1 + B_1)$		18
r	$E_1 - (B_1 + 2B_2)$		30
r	$E_1 - (A_1 + 2A_2)$	$E_1 - (2A_2 + A_3)$	3

There are 124 pairs $\{F, -F\}$ where F is of type (3.13) and the theta function vanishes to order r at F. $124 > 112$ where

$$112 = \binom{p+1}{3} + 2\binom{p+1}{2}, \qquad p = 7.$$

Case. $s = 3$, $t \geqslant 9$, $p = t + 1$.

Order of vanishing of the theta function	F	$-F$	number of pairs $\{F, -F\}$
$r + 1$	E_1	$E_1 - (A_1 + A_2 + A_3)$	1
r	$E_1 - (A_1 + A_2 + B_1)$	$E_1 - (A_3 + 2B_1)$	$3t$
r	$E_1 - (A_1 + B_1 + B_2)$		$3\binom{t}{2}$
r	$E_1 - (B_1 + B_2 + B_3)$		$\binom{t}{3}$
r	$E_1 - (2A_1 + B_1)$		$3t$
r	$E_1 - (2B_1 + B_2)$		$t(t-1)$
r	$E_1 - (A_1 + 2A_2)$	$E_1 - (2A_2 + A_3)$	3

There are

$$3t + 3\binom{t}{2} + \binom{t}{3} + 3t + t(t-1) + 3 = \binom{t+3}{3} + 2\binom{t+3}{2} - 3t - 4$$

pairs $\{F, -F\}$ of type (3.13) where $\theta(F) = 0$ r times. This number is larger than $\binom{p+1}{3} + 2\binom{p+1}{2}$.

Case. $s = 6$, $t = 6$, $p = 10$.

Order of vanishing of the theta function	F	$-F$	number of pairs $\{F, -F\}$
$r + 1$	E_1		
r	$E_1 - (A_1 + A_2 + A_3)$	$E_1 - (A_4 + A_5 + A_6)$	10
r	$E_1 - (B_1 + B_2 + B_3)$	$E_1 - (B_4 + B_5 + B_6)$	10
r	$E_1 - (A_1 + B_1 + B_2)$		90
r	$E_1 - (A_1 + A_2 + B_1)$		90
r	$E_1 - (C_1 + 2C_2)$		132

There are 332 pairs $\{F, -F\}$ where F is of type (3.13) and $\theta(F) = 0$ r times.

$$332 > 275 = \binom{p+1}{3} + 2\binom{p+1}{2}.$$

Case. $s = 6$, $t \geqslant 9$, $p = t + 4$.

Order of vanishing of the theta function	F	$-F$	number of pairs $\{F, -F\}$
$r+1$	E_1		
r	$E_1 - (A_1 + A_2 + A_3)$	$E_1 - (A_4 + A_5 + A_6)$	10
r	$E_1 - (B_1 + B_2 + B_3)$		$\binom{t}{3}$
r	$E_1 - (A_1 + B_1 + B_2)$		$6\binom{t}{2}$
r	$E_1 - (A_1 + A_2 + B_1)$		$15t$
r	$E_1 - (C_1 + 2C_2)$		$2\binom{t+6}{2}$

There are

$$10 + \binom{t}{3} + 6\binom{t}{2} + 15t + 2\binom{t+6}{2}$$

pairs $\{F, -F\}$ where F is of type (3.13) and $\theta(F) = 0$ r times. This number is

$$\binom{p+2}{3} + 2\binom{p+2}{2} - 10 > \binom{p+1}{3} + 2\binom{p+1}{2}.$$

Case. $s \geqslant 9$, $t \geqslant 9$, $p = s + t - 2$.

Order of vanishing of the theta function	F	$-F$	number of pairs $\{F, -F\}$
$r+1$	E_1		
r	$E_1 - (C_1 + C_2 + C_3)$		$\binom{p+2}{3}$
r	$E_1 - (2C_1 + C_2)$		$2\binom{p+2}{2}$

There are $\binom{p+2}{3} + 2\binom{p+2}{2}$ pairs $\{F, -F\}$ where F is of type (3.13) and $\theta(F) = 0$ r times. This number is greater than $\binom{p+1}{3} + 2\binom{p+1}{2}$.

All the cases of the lemma have been exhausted. Q.E.D.

The second aspect of the vanishing properties to be considered is some group theoretic properties of the (1/6)-periods. The whole set of (1/6)-periods where we have information about the vanishings of the theta functions is the coset $E_2 + \mathcal{G}$, which in this case equals $E_1 + \mathcal{G}$ since $s \equiv t \equiv 0 \pmod 3$. Let $\langle C_1 - C_2 \rangle$ be a cyclic subgroup of $\mathcal{G}$ of order three. Then the coset

$$E_1 - \langle C_1 - C_2 \rangle = \{E_1, E_1 - (C_1 + 2C_2), E_1 - (2C_1 + C_2)\}$$

is a set of zeros of the theta function of orders $r + 1$, r, and r, respectively. Moreover, none of these (1/6)-periods is a (1/2)-period. For if one were, then $2E_1 \equiv 0$, $C_1 + 2C_2$, or $2C_1 + C_2$. But $2E_1 \equiv \Sigma A_j \equiv \Sigma B_j \not\equiv 0$ by (3.7) and Lemma 3.1. We summarize these results in the following lemma. The results are also true for $r = 1$.

LEMMA 3.4. *Let $p = 3r + 1$, $r \geqslant 1$. Suppose W_p is a cyclic trigonal Riemann surface with $s > 0$. Then there is a cyclic subgroup of $\mathcal{G}$, $\langle \eta \rangle$, so that the theta function vanishes on the points of the coset $E_1 - \langle \eta \rangle$ to orders $r + 1$, r, and r. Moreover, none of the (1/6)-periods of the coset are (1/2)-periods.*

Case 2. $p \equiv 2 \pmod 3$. Let $p = 3r + 2$, $r \geqslant 1$. In this case the theta function vanishes at $E_1 - C_k$, $k = 1, 2, \ldots, p + 2$, to order $r + 1$. However, if $s = 2$ we have $2E_1 = A_1 + A_2$ or $(E_1 - A_1) + (E_1 - A_2) = 0$. This is compensated for by the existence of a half-period E_2 where the theta function vanishes to order r.

LEMMA 3.5. *Let $p = 3r + 2$, $r \geqslant 1$. If W_p is cyclic trigonal with $s > 2$ then there are $p + 2$ pairs $\{F, -F\}$ where F is of the form $E_1 - C_1$ and the theta function vanishes to order $r + 1$ at F. If $s = 2$ there are only $p + 1$ such pairs, but there is also a half-period E_2, where $3E_2 \equiv 3F$, and the theta function vanishes to order r at E_2.*

LEMMA 3.6. *Let $p = 3r + 2$, $r \geqslant 1$. Suppose W_p is a cyclic trigonal Riemann surface. Then there are $(1/6)$-periods $E_1 - C_k$, $k = 1, 2, \ldots, p + 2$, and cyclic subgroups of $\mathcal{G}$, $\langle \eta \rangle$, so that the theta function vanishes at the points of the coset $E_1 - C_k + \langle \eta \rangle$ to orders $r + 1$, $r + 1$, and r. Moreover, $\langle \eta \rangle$ can be chosen so that none of the $(1/6)$-periods in this coset is a $(1/2)$-period.*

PROOF. Fix any C_k and let $\eta = C_k - C_l$. The points of the coset are $E_1 - C_k$, $E_1 - C_l$, and $E_1 - (2C_k + 2C_l)$. The theta function vanishes as indicated. For any of these to be $(1/2)$-periods we must have $2E_1 \equiv C_k$, C_l, or $C_k + C_l$. This can only happen if $s = 2$ and $\{C_k, C_l\} = \{A_1, A_2\}$ so if $s = 2$ and C_k is one of the A's, C_l must be chosen different from the other A. Q.E.D.

Case 3. $p \equiv 3 \pmod 3$. Let $p = 3r + 3$, $r \geqslant 1$. $(1/6)$-periods of type (3.9) with the highest vanishings for theta are

$$E_1 - (C_k + C_l) \quad \text{or} \quad E_1 - (2C_k). \tag{3.14}$$

There are

$$\binom{p+2}{2} + \binom{p+2}{1} = \binom{p+3}{2}$$

such $(1/6)$-periods. Only if $s = 1$ and $C_k = A_1$ is a $(1/6)$-period of type (3.14) a $(1/2)$-period, namely $E_1 - 2A_1$. In all other cases the $(1/6)$-periods are not $(1/2)$-periods. The theta function vanishes to order $r + 1$ at $(1/6)$-periods of type (3.14). As in previous cases we want to count the number of pairs $\{F, -F\}$ where F is of type (3.14). Again we state the result in a form useful for the characterizations of §4.

LEMMA 3.7. *Let $p = 3r + 3$, $r \geqslant 1$. Suppose W_p is a cyclic trigonal Riemann surface. If $s > 1$ then there are more than $\binom{p+2}{2}$ pairs $\{F, -F\}$, where F is of type* (3.14) *and the theta function vanishes to order $r + 1$ at F. If $s = 1$ there are precisely $\binom{p+2}{2}$ such pairs and, additionally, there is a $(1/2)$-period E_2 so that $3E_2 \equiv 3F$ and $\theta(E_2) = 0$ $r + 1$ times.*

PROOF. Again we shall tabulate the results, being sure to account for all $\binom{p+2}{3}$ $(1/6)$-periods of type (3.14).

Case. $s = 1$, $t \geqslant 7$, $p = t - 1$.

Order of vanishing of the theta function	F	$-F$	number of pairs $\{F, -F\}$
$r + 1$	$E - 2A_1$	$E - 2A_1$	1
$r + 1$	$E - 2B_j$	$E - (A_1 + B_j)$	t
$r + 1$	$E - (B_i + B_j)$		$\binom{t}{2}$

There are

$$t + \binom{t}{2} = \binom{t+1}{2}$$

pairs $\{F, -F\}$ of (1/6)-periods and also the (1/2)-period $E_1 - 2A_1$. At all these points the theta function vanishes to order $r + 1$.

Case. $s = 4$, $t = 4$, $p = 6$.

Order of vanishing of the theta function	F	$-F$	number of pairs $\{F, -F\}$
2	$E - 2C_k$		8
2	$E_1 - (A_1 + A_2)$	$E_1 - (A_3 + A_4)$	3
2	$E_1 - (B_1 + B_2)$	$E_1 - (B_3 + B_4)$	3
2	$E_1 - (A_1 + B_1)$		16

Thus there are 30 such pairs and

$$30 > 28 = \binom{p+2}{2}, \qquad p = 6.$$

Case. $s = 4$, $t > 4$, $p = t + 2$.

Order of vanishing of the theta function	F	$-F$	number of pairs $\{F, -F\}$
$r + 1$	$E_1 - 2C_k$		$p + 2$
$r + 1$	$E_1 - (A_1 + A_2)$	$E_1 - (A_3 + A_4)$	3
$r + 1$	$E_1 - (B_1 + B_2)$		$\binom{p-2}{2}$
$r + 1$	$E_1 - (A_1 + B_1)$		$4(p - 2)$

Thus there are $\binom{p+2}{3} - 3$ such pairs and this number is greater than $\binom{p+2}{2}$.

Case. $s > 4$, $t \geqslant s$. In this case there are $\binom{p+3}{2}$ such pairs.

All cases have been exhausted. Q.E.D.

Now consider

$$E_1 - (C_k + C_l) + \langle C_k - C_l \rangle = \{E_1 - 2C_k, E_1 - (C_k + C_l), E_1 - 2C_l\}.$$

Only if $s = 1$ and C_k or $C_l = A_1$ are any of these (1/6)-periods a (1/2)-period. The theta function vanishes at each of these (1/6)-periods to order $r + 1$.

LEMMA 3.8. *Let* $p = 3r + 3$, $r \geq 1$. *Suppose* W_p *is a cyclic trigonal Riemann surface. Then there are* $(1/6)$-*periods* $E_1 - (C_k + C_l)$ *and cyclic subgroups of* $\mathcal{G}$, $\langle \eta \rangle$ ($\eta = C_k - C_l$) *so that the theta function vanishes to order* $r + 1$ *at the points of the coset* $E_1 - (C_k + C_l) + \langle \eta \rangle$. C_k, C_l *and* $\langle \eta \rangle$ *can be chosen so that none of the* $(1/6)$-*periods in the coset are* $(1/2)$-*periods.*

Case 4. $p \equiv 1 \pmod 3$ and $s = 0$. Let $p = 3r + 1$, $r \geq 2$. In this case, E_1 ($= E_2$) is a $(1/2)$-period and the theta function vanishes to order $r + 1$ at E_1. At $(1/6)$-periods of type

$$(3.15) \quad E_1 - (B_j + B_k + B_l) \quad \text{or} \quad E_1 - (2B_j + B_k) \quad \text{or} \quad E_1 - (2B_j + 2B_k + 2B_l)$$

the theta function vanishes to order r. Since

$$(E_1 - (B_j + B_k + B_l)) + (E_1 - (2B_j + 2B_k + 2B_l)) \equiv 0$$

and

$$(E_1 - (2B_j + B_k)) + (E_1 - (B_j + 2B_k)) \equiv 0,$$

we see there are

$$\binom{p+2}{3} + \binom{p+2}{2} = \binom{p+3}{3}$$

pairs $\{F, -F\}$ of $(1/6)$-periods where F is of type (3.15) and the theta function vanishes at F to order r.

LEMMA 3.9. *Let* $p = 3r + 1$, $r \geq 2$. *Suppose* W_p *is cyclic trigonal with* $s = 0$. *Then there are* $\binom{p+3}{3}$ *pairs* $\{F, -F\}$ *of* $(1/6)$-*periods so that* $\theta(F) = 0$ *r times. None of these* $(1/6)$-*periods are* $(1/2)$-*periods. Also there is a unique* $(1/2)$-*period* E_1 *so that* $\theta(E_1) = 0$ $r + 1$ *times and* $3E_1 \equiv 3F$.

Now consider subgroups of $\mathcal{G}$ of order nine as follows: $\langle B_1 - B_2, B_1 - B_3 \rangle$. The coset $E_1 - \langle B_1 - B_2, B_1 - B_3 \rangle$ is $\{E_1, E_1 \pm (B_1 + B_2 + B_3), E_1 \pm (B_1 + 2B_2), E_1 \pm (B_2 + 2B_3), E_1 \pm (B_3 + 2B_1)\}$. The theta function vanishes at the nine points of this coset to orders $r + 1, r, r, \ldots, r$. Such a coset will be called a 9-*coset.*

LEMMA 3.10. *There are* $\binom{p+2}{3}$ 9-*cosets.*

It should be pointed out that our discussion of high order vanishings of the theta function does not include a proof that we have accounted for *all* the high order vanishings (to order r or $r + 1$ as the case may be). However, in Lemmas 3.3, 3.5, 3.7, and 3.9 we have accounted for enough high order vanishings to characterize the existence of a cyclic trigonal automorphism, as we shall see in §4.

To see that we have, in fact, all the high order vanishings, one could make a complete table of all the vanishing properties, a task we have decided to postpone. Alternately, we can argue as follows.

If $\theta(F) = 0$ $k + 1$ times, then there is (by Lemma 2.4) a $g^k_{p-1} = kg^1_3 + D$, where $D = \Sigma \varepsilon_i c_i$, $\varepsilon_i = 0, 1, 2$, and $\Sigma \varepsilon_i = p - 1 - 3k$. If $p = 3r + 1$, then

$$g^{r-1}_{3r} = (r - 1)g^1_3 + c_i + c_j + c_k$$

and F (or $-F$) $= E_1 - (C_i + C_j + C_k)$ by (3.6). If $p = 3r + 2$, then

$$g^r_{3r+1} = rg^1_3 + c_i$$

and F (or $-F$) $= E_1 - C_i$ by (3.6). If $p = 3r + 3$, then

$$g^r_{3r+2} = rg^1_3 + c_i + c_j$$

and F (or $-F$) $= E_1 - (C_i + C_j)$ by (3.6). Thus, all the highest order vanishings are accounted for in the previous tables.

4. The characterizations for $p \geqslant 5$. In this section we show that a Riemann surface whose theta function has the vanishing properties derived in §3 and the nonvanishing properties of §2 is in fact cyclic trigonal. We consider the four cases separately. For each case we first prove the existence of a g^1_3. Then we show the g^1_3 has $p + 2$ 3-branch points; that is, it is cyclic trigonal. The cases are numbered as in §3.

4.1. Let $p = 3r + 1$, $r \geqslant 2$. We assume W_p has the vanishing and nonvanishing properties of a cyclic trigonal Riemann surface with $s \neq 0$; that is, W_p does not admit a half-canonical g^r_{3r}.

LEMMA 4.1.1. *W_p admits a g^1_3.*

PROOF. Assume W_p does not admit a g^1_3.

There are three $(1/6)$-periods in $J(W_p)$, E_1, $E_1 + \eta$, $E_1 + 2\eta$, none of which is a $(1/2)$-period, where the theta function vanishes to order $r + 1$, r, and r (Lemma 3.4).

Corresponding to the theta function vanishing at E_1 there is a g^r_{3r} on W_p which is not rg^1_3, by assumption. If g^r_{3r} were composite then there would be a two-sheeted cover $W_p \to W_q$ and a complete $g^r_{(3r-\varepsilon)/2}$ on W_q which lifts to the nonfixed points of g^r_{3r}. By Clifford's theorem, $g^r_{(3r-\varepsilon)/2}$ is not special and so by the Riemann-Roch theorem $q = (r - \varepsilon)/2$. Thus $6q = 3r - 3\varepsilon = p - 1 - 3\varepsilon$. But the nonvanishing properties of the theta function (Lemma 2.7) exclude this value of q. Consequently g^r_{3r} must be simple.

On the smooth three-sheeted cover W_{3p-2} of W_p corresponding to the subgroup $\langle \eta \rangle$ in $J(W_p)$ there is a g^{3r}_{9r} obtained by lifting the three linear series corresponding to the theta function vanishing at E_1, $E_1 + \eta$, $E_1 + 2\eta$ (Lemma 2.6). g^{3r}_{9r} must be simple.

We assert that g^{3r}_{9r}, which is $(1/6)$-$3K$, is not $(1/2)$-K. On W_p there is a $(1/6)$-$3K$ h^r_{3r} so that $g^r_{3r} + h^r_{3r} \equiv K$. If g^{3r}_{9r} is $(1/2)$-K then g^r_{3r} and h^r_{3r} lift to equivalent divisors on W_{3p-2}. Thus h^r_{3r} corresponds to one of the periods in $E_1 + \langle \eta \rangle$, which must, in fact, be E_1, since the theta function vanishes to order $r + 1$ there. This means E_1 is a $(1/2)$-period, a contradiction.

Thus g^{3r}_{9r} is not $(1/2)$-K. By Lemmas 2.3 and 2.8, W_{3p-2} admits a g^1_4. W_p admits a g^1_3 after all. Q.E.D.

REMARK. One can find a similar proof without going to W_{3p-2}, but eliminating the possibility of a g^1_4 on W_p turns out to be extremely tedious.

The referee points out that in applying Lemmas 2.3 and 2.8, a remark excluding the possibility of more than one g^1_4 on W_{3p-2} is in order. Two g^1_4's on W_{3p-2} would imply W_{3p-2} is (0-H) or (1-H) since $3p - 2 \geqslant 19$. Therefore W_p would also be (0-H) or (1-H). But this is excluded by the nonvanishing properties.

Now we show W_p is cyclic trigonal by showing that the covering $W_p \to \mathbf{P}'$ given by g_3^1 has $p + 2$ 3-branched points.

Consider a pair of (1/6)-3K linear series, g_{3r}^{r-1}, h_{3r}^{r-1}, whose sum is canonical, corresponding to the theta function vanishing at a pair $\{F, -F\}$ to order r. By Lemma 2.4 at least one of them is $(r-1)g_3^1 + x_1 + x_2 + x_3$ for three points x_1, x_2, x_3 on W_p. Since one of the two, g_{3r}^r or h_{3r}^r, is rg_3^1 and $3g_{3r}^r \equiv 3h_{3r}^r \equiv 3g_{3r}^{r-1}$, we see that

$$3(x_1 + x_2 + x_3) \equiv 3g_3^1 = g_9^{3+\varepsilon}.$$

If $\varepsilon \geqslant 1$ then W_p admits a simple g_9^4 and, by Castelnuovos' theorem, (Lemma 2.2), it follows that $p = 7$. But if $p = 7$, $g_9^4 + g_3^1 \equiv K$ so $4g_3^1 = K$. $2g_3^1$ is (1/2)-K, a possibility excluded by hypothesis ($s > 0$). Consequently, $3g_3^1 \equiv g_9^3$, which is complete and composite.

Lemma 4.1.2. $3x_1 \equiv 3x_2 \equiv 3x_3 \equiv g_3^1$.

Proof. *Case* 1. Suppose $x_1 \neq x_2 \neq x_3 \neq x_1$. $3x_1 + 3x_2 + 3x_3$ is the sum of 3 divisors in g_3^1 and $x_1 + x_2 + x_3$ is not a divisor in g_3^1 since g_{3r}^{r-1} is complete. Suppose $x_1 + x_2$ is in one divisor D_1 of g_3^1 and x_3 is in D_2. Then $D_1 + D_2 \subset 3(x_1 + x_2 + x_3)$, and it follows that $3x_3 = D_2$ and $3(x_1 + x_2) = 2D_1$. By relabeling if necessary we see that $2x_1 + x_2 = D_1$, so $x_1 + 2x_2 = D_1$ and $x_1 = x_2$, a contradiction. Consequently, x_1, x_2 and x_3 must lie in three distinct divisors of g_3^1. The lemma now follows in this case.

Case 2. Suppose $x_1 = x_2 \neq x_3$. Then $6x_1 + 3x_3 = 3g_3^1$. If x_1 and x_3 are in the same divisor of g_3^1, we see that $g_3^1 = 2x_1 + x_3$, a contradiction. Consequently, x_1 and x_3 are in different divisors of g_3^1. The lemma now follows in this case.

Case 3. Suppose $x_1 = x_2 = x_3$. This is seen to be impossible. Q.E.D.

Theorem 4.1.3. *W_p is cyclic trigonal.*

Proof. Every vanishing of the theta function at a pair of (1/6)-periods $\{F, -F\}$ to order r gives rise to either $(r-1)g_3^1 + x_1 + x_2 + x_3$ where $3x_1 \equiv 3x_2 \equiv 3x_3 \equiv g_3^1$, or $(r-1)g_3^1 + 2x_1 + x_2$ where $3x_1 \equiv 3x_2 \equiv g_3^1$. Each x_i is a 3-branch point of g_3^1. Notice that the existence of 7 (= 1 + 6) such pairs $\{F, -F\}$ implies only the existence of 3 such x_i's. The existence of 16 (= 4 + 12) such pairs $\{F, -F\}$ implies the existence of 4 such x_i's. In general if we have $\binom{n}{3} + 2\binom{n}{2}$ such pairs $\{F, -F\}$ we can assume the existence of only n such point x_i. But if we have *more* than $\binom{n}{3} + 2\binom{n}{2}$ such pairs, there must be more than n such x_i's. By Lemma 3.3 there are more than $\binom{p+1}{3} + 2\binom{p+1}{2}$ such pairs. So g_3^1 has $p + 2$ 3-branch points. Q.E.D.

4.2. $p = 3r + 2$, $r \geqslant 1$. We assume W_p has the vanishing and nonvanishing properties of a cyclic trigonal Riemann surface.

Lemma 4.2.1. *W_p admits a g_3^1.*

Proof. We assume W_p does not admit a g_3^1.

There are three (1/6)-periods in $J(W_p)$, $E_1 = C_k + \langle \eta \rangle$ ($\eta = C_k - C_l$) where the theta function vanishes to order $r + 1$, $r + 1$ and r, respectively. We can choose this coset so that none of the periods is a (1/2)-period (Lemma 3.6).

Corresponding to the theta function vanishing at $E_1 - C_k$ there is a g^r_{3r+1} which is not $rg^1_3 + x$. If g^r_{3r+1} is composite, then one sees there is a two-sheeted cover $W_p \to W_q$ and a $g^r_{(3r+1-\varepsilon)/2}$ on W_q which is nonspecial if $r \geqslant 2$ (by Clifford's theorem). By the Riemann-Roch theorem, $q = (r + 1 - \varepsilon)/2$. Thus $6q = 3r + 3 - 3\varepsilon = p + 1 - 3\varepsilon$. If p is even, r is even, so ε is odd; that is, if p is even, $6q \leqslant p - 2$. If p is odd. $6q \leqslant p + 1$. But the nonvanishing properties of the theta function exclude both these possibilities (Lemma 2.7). Thus if $r \geqslant 2$, g^r_{3r+1} is simple. (We will handle the case $r = 1$ shortly.)

Now on the smooth 3-sheeted cover W_{3p-2} of W_p corresponding to the subgroup $\langle \eta \rangle$ in $J(W_p)$, there is a g^{3r+1}_{9r+3} obtained by lifing the three linear series corresponding to the theta function vanishing at $E_1 - C_k + \langle \eta \rangle$. g^{3r+1}_{9r+3} is simple if $r \geqslant 2$ since g^r_{3r+1} is simple (Lemma 2.6).

If $r = 1$ we show that g^4_{12} on W_{13} is simple. If not, it cannot be $4g^1_3$ so W_{13} is a two-sheeted cover of W_q admitting a $g^4_{(12-\varepsilon)/2}$. Thus $q = 0, 1$, or 2. The covering $W_{13} \to W_q$ is strongly branched.

This implies W_5 is a two-sheeted cover of $W_{q'}$, over which W_q is a 3-sheeted cover. This implies $q' = 0$ or 1, a contradiction of the nonvanishing properties of the theta function for W_5.

Now we assert that g^{3r+1}_{9r+3} is not $(1/2)$-K. Since g^r_{3r+1} is $(1/6)$-K, there is another $(1/6)$-$3K$ h^r_{3r+1} so that $g^r_{3r+1} + h^r_{3r+1} = K$. If g^{3r+1}_{9r+3} were $(1/2)$-K then g^r_{3r+1} and h^r_{3r+1} lift to equivalent divisors on W_{3p-2}. This implies $2(E_1 - C_k) \in \langle \eta \rangle$, so one of the periods $E_1 - C_k + \langle \eta \rangle$ is a half-period, contradiction (Lemma 3.6). Thus g^{3r+1}_{9r+3} is simple and not $(1/2)$-K. So W_{3p-2} admits a g^1_3 and so does W_p after all (Lemmas 2.3 and 2.8 and the referee's remarks following Lemma 4.1.1). Q.E.D.

Each pair $\{F, -F\}$ ($6F \equiv 0$, $\theta(F) = 0$ $r + 1$ times) gives rise to a pair g^r_{3r+1}, h^r_{3r+1} whose sum is canonical and at least one of these is $rg^1_3 + x_i$ by Lemma 2.4. Since there are at least $p + 1$ such pairs $\{F_i, -F_i\}$ such that $3F_i \equiv 3F_j$, we see that

$$3(rg^1_3 + x_i) \equiv 3(rg^1_3 + x_j) \quad \text{or} \quad 3x_i \equiv 3x_j \equiv g^1_3$$

since g^1_3 is unique.

THEOREM 4.2.2. *W_p is cyclic trigonal.*

PROOF. By consulting the table of vanishings (§3) we see that if $s \geqslant 5$ there are $p + 2$ pairs $\{F_i - F_i\}$. Each x_i is a 3-branch point for g^1_3 so the proof is complete when there are $p + 2$ such pairs.

If there are only $p + 1$ such pairs ($s = 2$) we must find the $(p + 2)$nd 3-branch point for g^1_3. We do this by showing that there is a pair $\{F, -F\}$ where the complementary linear series corresponding to F and $-F$ *both* give rise to a 3-branch point.

There is a $(1/2)$-period E_2 so that $\theta(E_2) = 0$ r times and $3E_2 \equiv 3F_i$ for all pairs $\{F_i, -F_i\}$. This half-period gives rise to a $(1/2)$-K linear series g^{r-1}_{3r+1} which is $(r - 1)g^1_3 + D_4$, D_4 a divisor of degree four. Since $3g^{r-1}_{3r+1} = (3r + 1)g^1_3$, we see that $3D_4 \equiv 4g^1_3 \equiv g^{4+\varepsilon}_{12}$. If $\varepsilon \geqslant 1$ then W_p admits a simple g^5_{12}, so $p \leqslant 10$ by Castelnuovos' inequality.

Assume $p \geqslant 11$ ($r \geqslant 3$).

$$D_4 = t + x + y + z \quad \text{and} \quad 3t + 3x + 3y + 3z \equiv 4g_3^1 = g_{12}^4.$$

The latter series is complete and composite. We show that (by relabeling if necessary) $t = x$ and $y = z$ and $3t \equiv 3y \equiv g_3^1$. First observe that no divisor of g_3^1 can be made up of three of the points of D_4 since g_{3r+1}^{r-1} is complete.

Now suppose the four points of D_4 lie in four distinct divisors of g_3^1. Then one sees that $3t \equiv 3x \equiv 3y \equiv 3z$. But

$$2g_{3r+1}^{r-1} = K \quad \text{or} \quad 2t + 2x + 2y + 2z + (2r-2)g_3^1 = K.$$

Since two of the points of $3t$ are in $2t + 2x + 2y + 2z$, a special divisor, and $(2r-2)g_3^1$ is without fixed points, we see that t is a point of $2x + 2y + 2z$, a contradiction.

Next suppose $t + x$ lies in a divisor of g_3^1. Since $y + z$ cannot contain a point of this divisor we see that $2t + 2x$ must contain a divisor of g_3^1. Suppose $2t + x$ is such a divisor. Then $t + 2x + 3y + 3z \equiv 3g_3^1$. It follows that $t + 2x$ is also a divisor in g_3^1, so $t = x$, $3t \equiv g_3^1$. Then $3y + 3z \equiv 2g_3^1$. By an entirely analogous argument we see that $y = z$ and $3y \equiv g_3^1$. Thus

$$2(r-1)g_3^1 + 4t + 4y = 2rg_3^1 + t + y \equiv K.$$

$rg_3^1 + t$ and $rg_3^1 + y$ are both $(1/6)$-$3K$ linear series corresponding to the same pair $\{F, -F\}$ where the theta function vanishes $r + 1$ times. Thus we have found the $(p+2)$nd 3-branch point for g_3^1. This completes the proof for $p \geqslant 11$ when there are only $p + 1$ pairs $\{F, -F\}$.

Now assume $r = 2$, $p = 8$ and there are only nine pairs $\{F, -F\}$. If in the previous part of the proof of this section $4g_3^1 = g_{12}^4$ is complete and composite, we can complete the proof as above (in fact, this case does not occur). If, however, $4g_3^1 = g_{12}^5$ then $K = g_{12}^5 + t + y = 4g_3^1 + t + y$. We know $3K \equiv 14g_3^1$ so $3t + 3y = 2g_3^1 = g_6^2$, complete and composite. It follows that $3t \equiv 3y \equiv g_3^1$. Also $t \neq y$ since there is no $(1/2)$-K g_7^2. Consequently $2g_3^1 + t$ and $2g_3^1 + y$ give rise to the tenth 3-branch point for g_3^1.

Finally, if $r = 1$ then $2g_3^1 = g_6^2$, which is special. Consequently $K \equiv 2g_3^1 + t + y$ and the proof is completed as in the previous paragraph. Q.E.D.

4.3. $p = 3r + 3$, $r \geqslant 1$. We assume W_p has the vanishing and nonvanishing properties of a cyclic trigonal Riemann surface.

LEMMA 4.3.1. *W_p admits a g_3^1.*

PROOF. Assume W_p does not admit a g_3^1.

We can find $(1/6)$-periods in $J(W_p)$, $E_1 - 2C_k + \langle \eta \rangle$ ($\eta = C_k - C_l$), none of which is a $(1/2)$-period, where the theta function vanishes to order $r + 1$, $r + 1$, and $r + 1$ (Lemma 3.8). On a smooth, cyclic, 3-sheeted cover of W_p, W_{3p-2}, corresponding to $\langle \eta \rangle$, the three linear series on W_p corresponding to the vanishings lift to a $1/6$-$3K$ g_{9r+6}^{3r+2}. We show that g_{9r+6}^{3r+2} is simple. If it were composite it cannot be $(3r+2)g_3^1$ since W_p does not admit a g_3^1. Consequently W_{3p-2} is a two-sheeted cover of $W_{q'}$ which admits a simple $g_{(9r+6-\varepsilon)/2}^{3r+2}$. As before $q' = (3r + 2 - \varepsilon)/2$ and

$W_{3p-2} \to W_{q'}$ is strongly branched. Consequently, the automorphism of period three on W_{3p-2} descends to $W_{q'}$, $W_{q'} \to W_q$ is a smooth cyclic 3-sheeted cover, and $W_p \to W_q$ is two-sheeted. But

$$6q - 6 = 2q' - 2 = 3r - \varepsilon = p - 3 - \varepsilon,$$

or $6q = p + 3 - \varepsilon$. If p is even, r is odd so ε must be at least 3; that is, $6q \leqslant p$. If p is odd we have $6q \leqslant p + 3$. But the nonvanishing properties of W_p (Lemma 2.7) exclude these two possibilities (just barely). Consequently g^{3r+2}_{9r+6} is simple on W_{3p-2}.

Now g^{3r+2}_{9r+6} is not half-canonical by an argument analogous to that in previous sections.

Since g^{3r+2}_{9r+6} is not half-canonical it follows as before that W_{3p-2} admits a g^1_3. Consequently W_p does also. Q.E.D.

For each pair of (1/6)-periods (not (1/2)-K) $\{F_i, -F_i\}$ where the theta function vanishes to order $r + 1$, there is a pair of (1/6)-$3K$ linear series whose sum is canonical, $g^r_{3r+2} + h^r_{3r+2} \equiv K$. We want to show that for at least one pair, $3g^r_{3r+2} \equiv (3r + 2)g^1_3 \equiv 3h^r_{3r+2}$. But the proof of Lemma 4.3.1 shows that on a suitable 3-sheeted cover of W_p. W_{3p-2}, the lift of g^r_{3r+2} or h^r_{3r+2} must be $(3r + 2)g^1_3 = g^{3r+2}_{9r+6}$ by Lemma 2.4. Dropping this equation to W_p gives $(3r + 2)g^1_3 = 3g^r_{3r+2}$. Having proven this for one pair $\{F_i, -F_i\}$, it holds for all linear series corresponding to all pairs since $3F_i \equiv 3F_j$.

THEOREM 4.3.2. *W_p is cyclic trigonal.*

PROOF. Every vanishing of the theta function at a pair $\{F_i, -F_i\}$, $6F_i \equiv 0$, to order $r + 1$ gives rise to either $rg^1_3 + x + y$ or $rg^1_3 + 2x$ where $3x \equiv 3y \equiv g^1_3$ since $3(x + y) \equiv 2g^1_3$ or $6x \equiv 2g^1_3$. Each such x or y is a 3-branch point for g^1_3. If there are three such pairs $\{F_i, -F_i\}$ there are at least 2 such 3-branch points. If there are $\binom{n+1}{2}$ such pairs, then there are at least n such 3-branch points.

Now if $s \geqslant 4$ we know there are more than $\binom{p+2}{2}$ such pairs; consequently the g^1_3 has more than $p + 1$ 3-branch points.

If $s = 1$ there are precisely $\binom{p+2}{2}$ such pairs, so we look for the $(p + 2)$nd 3-branch point elsewhere. There is in this case a half period E_2 at which the theta function vanishes $r + 1$ times and $3E_2 \equiv 3F_i$. The corresponding (1/2)-K linear series must be $rg^1_3 + x + y$, where $3x \equiv 3y \equiv g^1_3$ and $2x + 2y$ is special. It follows that $x = y$ and $3x \equiv g^1_3$. This is the additional 3-branch point for g^1_3. Q.E.D.

4.4. Let $p = 3r + 1$, $r \geqslant 2$. We assume W_p has the vanishing and nonvanishing properties of a cyclic trigonal Riemann surface with $s = 0$. W_p admits a unique (1/2)-K g^r_{3r}.

LEMMA 4.4.1. *W_p admits a g^1_3.*

PROOF. There is a 9-group $\langle \eta, \nu \rangle$ ($\eta = C_k - C_l$, $\nu = C_k - C_m$) so that

$$\theta(E_1 + \varepsilon_1\eta + \varepsilon_2\nu) = 0$$

r times ($\varepsilon_1, \varepsilon_2 = 0, 1, 2$, not both zero) while $\theta(E_1) = 0$ $r + 1$ times. On a smooth 9-sheeted cover of W_p, W_{9p-8}, corresponding to $\langle \eta, \nu \rangle$ there is a (1/2)-K g^{9r}_{27r}. Since $r \geqslant 2$ and $9p - 8 \geqslant 55$, by Lemma 2.9 W_{9p-8} must admit a g^1_3, a g^1_4 or a γ^1_3. The g^1_4 is excluded by Lemma 2.8, so W_{9p-8} has a 3-sheeted map to $W_{q'}$ where $q = 0$ or 1. The map is strongly branched in either case so the group $Z_3 \times Z_3$ descends to act on $W_{q'}$. Since $W_{9p-8} \to W_{q'}$ is not an intermediate covering of $W_{9p-8} \to W_p$, the entire group acts nontrivially on $W_{q'}$. Then $q = 1$ since the group cannot act on $\mathbf{P}^1$; that is, W_{9p-8} admits a γ_3^1.

Now suppose W_p does not admit a g_3^1.

The quotient of W_1' $(q = 1)$ by $Z_3 \times Z_3$ has genus one. Thus we obtain a configuration of coverings of Riemann surfaces as in Diagram I.

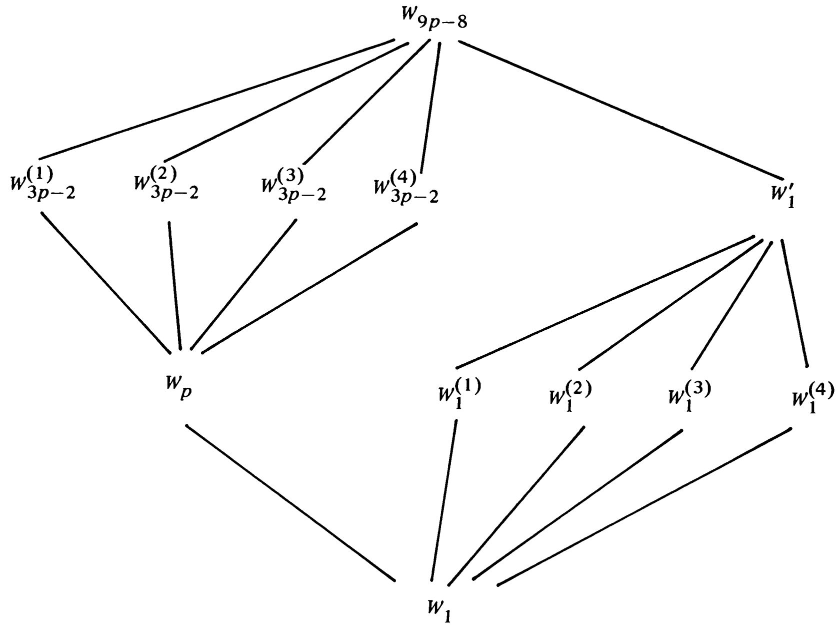

DIAGRAM I

We assert that g_{3r}^r on W_p is lifted from a g_r^{r-1} on W_1. Once we know this, then by Lemma 2.10 we know W_p admits a complete $(1/2)$-K h_{3r}^{r-1}. This contradicts the nonvanishing properties of W_p (Lemma 2.11) and the proof will be concluded.

To prove the assertion we remind ourselves that $g_{27r}^{9r} = |g_{27r}^{9r-1}|$ on W_{9p-8}, where g_{27r}^{9r-1} is the lift from W_1' of a g_{9r}^{9r-1}. The lift of g_{3r}^r from W_p to W_{9p-8} is a g_{27r}^r, also included in g_{27r}^{9r}. By counting dimensions, we see there is a divisor D common to g_{27r}^{9r-1} and g_{27r}^r. D is lifted to W_{9p-8} from W_p and from W_1'; consequently, it is the lift from W_1 of a divisor D_0 of degree r. On W_1, $|D_0| = g_r^{r-1}$ and D_0 lifted to W_p is in g_{3r}^r. This proves the assertion. Q.E.D.

(We thank the referee for greatly clarifying the proof of this lemma.)

Notice that the above proof only concludes that the genus of W_1 is zero. W_1' must still be a torus since it admits a $Z_3 \times Z_3$ of automorphisms. Consequently, everything in Diagram I is correct except the genera of W_1 and three of the $W_1^{(i)}$. Since we will need this fact shortly, we include Diagram II which exhibits the correct genera.

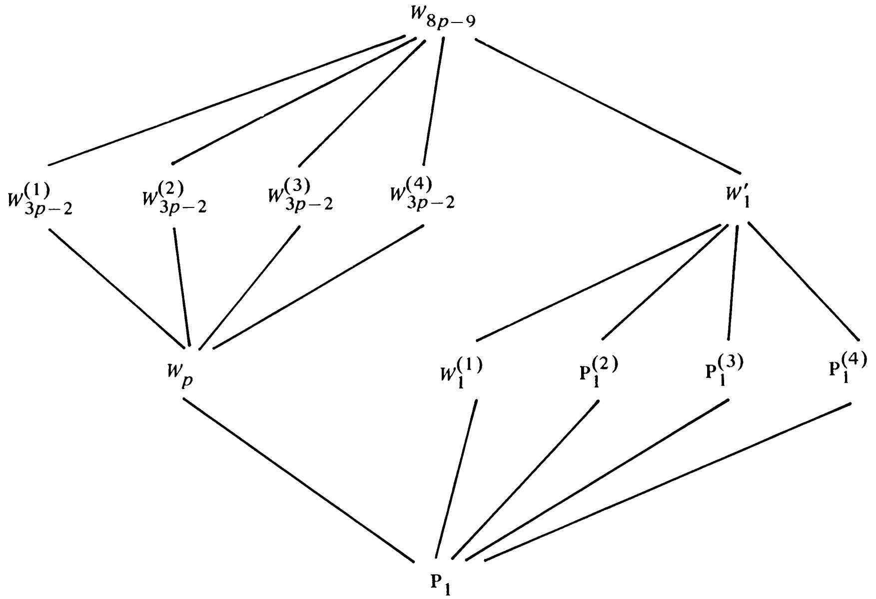

DIAGRAM II

Every pair of (1/6)-periods $\{F, -F\}$ where the theta function vanishes to order r leads to a (1/6)-3K linear series $(r-1)g_3^1 + x_1 + x_2 + x_3$. Since $3F \equiv 3E_1$ and $g_{3r}^r = rg_3^1$, we see that $3(x_1 + x_2 + x_3) \equiv 3g_3^1 = g_9^3$, complete and composite, unless $p = 7$. It follows that $3x_1 \equiv 3x_2 \equiv 3x_3 = g_3^1$ if $p > 7$, and from this, by elementary counting, we can derive the $p + 2$ 3-branch points for g_3^1 as before. For $p = 7$ this does not seem to work, so we use an argument that will work for all $r \geqslant 2$.

For each 9-coset we obtain a surface W_{9p-8} and a configuration of covering as in Diagram II. We show that each 9-coset corresponds to a unique triple of 3-branch points in g_3^1. Since there are $\binom{p+2}{3}$ 9-cosets, elementary counting now insures the existence of $p + 2$ 3-branch points for g_3^1.

Consider the subdiagram

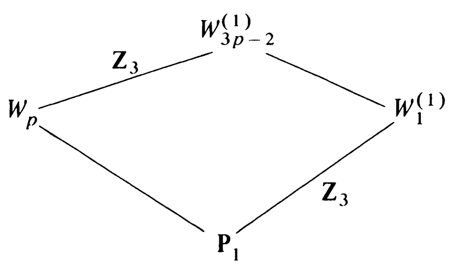

DIAGRAM III

The Z_3's indicate cyclic covers. $W_1^{(1)} \to \mathbf{P}_1$ is branched over 3 points in $\mathbf{P}_1$, say δ_1, δ_2 and δ_3. But the cover $W^{(1)}_{3p-2} \to W_p$ is smooth. This implies $W_p \to \mathbf{P}_1$ has three 3-branch points over δ_1, δ_2, and δ_3. $W^{(1)}_{3p-2}$ is uniquely determined by the cover $W_p \to \mathbf{P}_1$ and the points δ_1, δ_2, and δ_3 since it is the minimal covering of the covers

$W_p \to \mathbf{P}_1$ and $W_1^{(1)} \to \mathbf{P}_1$. Also, the whole of the configuration of coverings in Diagram II is determined by these two coverings since the 3-sheeted coverings $\mathbf{P}_1^{(i)} \to \mathbf{P}_1$ are determined by pairs of the three points δ_1, δ_2, and δ_3. This completes the proof.

THEOREM 4.4.2. *W_p is cyclic trigonal.*

REMARK. The proof shows that $W_{9p-8} \to W_1'$ is cyclic so W_{9p-8} admits an elementary abelian group of automorphisms of order 27.

5. One case for genus four. In this section we assume W_4 admits two distinct linear series, g_3^1 and h_3^1, whose sum in canonical. We characterize the cases when one or both of the trigonal coverings is cyclic.

First we assume W_4 is the Riemann surface for the equation

$$y^3 = \prod_{i=1}^{3} (x - \alpha_i) \prod_{j=1}^{3} (x - \beta_j)^2,$$

and we tabulate the vanishing properties of the theta function.

Order of vanishing of the theta function	F	$-F$	number of pairs $\{F, -F\}$
2	E_1	$E_1 - (A_1 + A_2 + A_3)$	1
1	$E_1 - (A_1 + 2A_2)$	$E_1 - (2A_2 + A_3)$	3
1	$E_1 - (B_1 + 2B_2)$	$E_1 - (2B_2 + B_3)$	3
1	$E_1 - (A_1 + A_2 + B_1)$	$E_1 - (A_3 + 2B_1)$	9
1	$E_1 - (A_1 + B_1 + B_2)$	$E_1 - (2A_1 + B_3)$	9

Thus there is one pair $\{F, -F\}$ when $\theta(F) = 0$ twice and there are twenty-four pairs where $\theta(F) = 0$ once.

Let us assume h_3^1 is the cyclic cover and

$$g_3^1 = |a_1 + a_2 + a_3| = |b_1 + b_2 + b_3|.$$

Thus $u(h_3^1) + \kappa \equiv -E_1$ and

$$u(g_3^1) + \kappa \equiv -(E_1 - (A_1 + A_2 + A_3)) \equiv -(E_1 - (B_1 + B_2 + B_3)).$$

The linear series of dimension zero are combinations of three of the a's and b's; e.g., if $g_3^0 = |a_1 + 2b_3|$ then

$$u(g_3^0) + \kappa \equiv -(E_1 - (A_1 + 2B_3)).$$

The following observations will be useful in the characterizations to follow, so we gather them together in a lemma.

LEMMA 5.1. *There are three kinds of g_3^0's corresponding to the $(1/6)$-periods in the preceding table.*

(i) *g_3^0's so that*

(a) *there is an h_3^0 and $g_3^0 + h_3^0 \equiv 2g_3^1$;*

(b) *there is a k_3^0 and $g_3^0 + k_3^0 \equiv 2h_3^1$.*

(ii) g_3^0'*s so that*

(a) *there is an* h_3^0 *and* $g_3^0 + h_3^0 \equiv 2g_3^1$;

(b) *there does not exist a* k_3^0 *so that* $g_3^0 + k_3^0 \equiv 2h_3^1$

(iii) g_3^0'*s so that*

(a) *there does not exist an* h_3^0 *so that* $g_3^0 + h_3^0 \equiv 2g_3^1$;

(b) *there is a* k_3^0 *so that* $g_3^0 + k_3^0 \equiv 2h_3^1$.

PROOF. (i) Let $g_3^0 = |a_1 + 2a_2|$, $h_3^0 = |a_1 + 2a_3|$ and $k_3^0 = |2a_1 + a_2|$.

(ii) Let $g_3^0 = |a_1 + a_2 + b_1|$ and $h_3^0 = |a_3 + b_2 + b_3|$.

(iii) Let $g_3^0 = |a_1 + 2b_1|$ and $k_3^0 = |2a_1 + b_1|$. Q.E.D.

REMARK. The nonexistence of Lemma 5.1 (ii)(b) and (iii)(a) is reflected in nonvanishings. For instance, in (ii)(b) above, the $(1/6)$-period corresponding to the missing k_3^0 is one where the theta function does not vanish.

If W_4 admits two cyclic trigonal coverings there will be two sets of vanishings at $(1/6)$-periods as in the table. The only $(1/6)$-periods common to the two tables will be the $(1/6)$-periods where the theta function vanishes twice.

Now assume the theta function for W_4 has vanishings at $(1/6)$-periods as in the table. We will show that W_4 is a cyclic trigonal Riemann surface.

W_4 admits $(1/6)$-$3K$ linear series g_3^1, h_3^1 and a collection of g_3^0's, and the triples of all of them are linearly equivalent. $2g_3^1$ and $2h_3^1$ are both g_6^2's which are complete and composite. The various g_3^0's will be denoted, as need be, by h_3^0, k_3^0, $1_3^0, \ldots$.

LEMMA 5.2. *Suppose* $g_3^0 + h_3^0 \equiv 2g_3^1$. *Then one of the following cases must hold.*

(i) $g_3^0 = 2x + y$, $h_3^0 = x + 2y$, $x \neq y$, *and* $3x \equiv 3y \equiv g_3^1$.

(ii) $g_3^0 = 2x + y$, $h_3^0 = y + 2z$, $x \neq y \neq z \neq x$, $g_3^1 = |x + y + z|$ *and* $h_3^1 \equiv 3x \equiv 3y \equiv 3z$.

(iii) $g_3^0 = x_1 + x_2 + x_3$; $h_3^0 = y_1 + y_2 + y_3$, *the* x'*s and* y'*s are six distinct points on* W_4, $g_3^1 \equiv x_1 + y_2 + y_3 \equiv y_1 + x_2 + x_3$ (*by relabelling if necessary*) *and* $h_3^1 \equiv 3x_1 \equiv 3x_2 \equiv 3x_3 \equiv 3y_1 \equiv 3y_2 \equiv 3y_3$ (*and so* W_4 *is cyclic trigonal*).

PROOF. First we show $g_3^0 = 3x$ is impossible.

If $g_3^0 = 3x$ and $h_3^0 = y + z + t$, then $2g_3^1 \equiv 3x + y + z + t$. By relabelling if necessary we see that $g_3^1 \equiv x + y + z \equiv 2x + t$. $t \neq x$ since $g_3^1 \neq g_3^0$. But $3g_3^1 \equiv 3g_3^0$ so $9x \equiv 6x + 3t$ or $3x \equiv 3t$. Thus g_3^0 has dimension greater than zero, a contradiction.

Now assume $g_3^0 = 2x + y$, $x \neq y$, $h_3^0 = z + s + t$. Thus $2g_3^1 \equiv 2x + y + z + s + t$. There are two cases.

(i) $g_3^1 \equiv 2x + z$ (by relabelling if necessary) $\equiv y + s + t$. Then

$$3g_3^1 \equiv 6x + 3z \equiv 3g_3^0 \equiv 6x + 3y \quad \text{and} \quad 3z \equiv 3y.$$

Since $y \neq z$ we see that $3z \equiv 3y \equiv k_3^1$. Suppose $k_3^1 = h_3^1$. Then

$$3g_3^1 \equiv 6x + 3z \equiv 3h_3^1 \equiv 9z, \qquad 2h_3^1 \equiv 6z \equiv 6x,$$

so $h_3^1 \equiv 3x$.

But $g_3^1 = 2x + z$ so h_3^1 and g_3^1 have $2x$ in common. This is impossible ($p = 4$). Consequently $3z \equiv 3y \equiv g_3^1$. It follows that $x = z$ and $y = s = t$. This is case (i) in the statement of the lemma.

(ii) $g_3^1 \equiv x + z + s \equiv x + y + t$ (by relabelling if necessary). We can assume without loss of generality that $z = y$ and $s = t$. Then

$$3g_3^0 \equiv 6x + 3y \equiv 3g_3^1 \equiv 3x + 3y + 3t \quad \text{and} \quad 3x \equiv 3t \equiv k_3^1.$$

If $k_3^1 = g_3^1$ then $x = z = s$, a contradiction. Consequently $h_3^1 \equiv 3x \equiv 3t$. $3h_3^1 \equiv 9x \equiv 3g_3^0 \equiv 6x + 3y$ so

$$h_3^1 \equiv 3x \equiv 3y \equiv 3t \quad \text{and} \quad g_3^1 \equiv x + y + t.$$

This is case (ii) of the statement of the lemma.

(iii) Finally suppose $g_3^0 = x_1 + x_2 + x_3$, three distinct points. Then $h_3^0 = y_1 + y_2 + y_3$, again three distinct points. (Otherwise, by cases (i) and (ii) g_3^0 could not have three distinct points.) Now $2g_3^1 = x_1 + x_2 + x_3 + y_1 + y_2 + y_3$. By relabeling if necessary we can assume $g_3^1 \equiv x_1 + y_2 + y_3 \equiv y_1 + x_2 + x_3$. Since $3g_3^1 \equiv 3h_3^0$, we see that $3x_1 \equiv 3y_1$. Since $x_1 \neq y_1$ it follows that $3x_1 \equiv h_3^1$. Consequently $2h_3^1 \equiv 3x_2 + 3x_3 \equiv 3y_2 + 3y_3$ and it follows that $h_3^1 \equiv 3x_2 \equiv 3x_3 \equiv 3y_2 \equiv 3y_3$. Now by examining all cases it follows by arguments already used that the six points $x_1, x_2, \ldots, y_3$ are all distinct. Q.E.D.

LEMMA 5.3. *Suppose* $g_3^0 + h_3^0 \equiv 2g_3^1$, $g_3^0 + k_3^0 \equiv K$, *and* $h_3^0 + l_3^0 \equiv K$; *therefore* $k_3^0 + l_3^0 \equiv 2h_3^1$. *Then*

(a) *If g_3^0 and h_3^0 satisfy case* (i) *of Lemma* 5.2, *then k_3^0 and l_3^0 satisfy cases* (ii) *or* (iii) *of Lemma* 5.2 (*with g_3^1 and h_3^1 interchanged*).

(b) *If g_3^0 and h_3^0 satisfy case* (ii) *of Lemma* 5.2 *then k_3^0 and l_3^0 satisfy case* (i) *of Lemma* 5.2 (*with g_3^1 and h_3^1 interchanged*).

(c) *If g_3^0 and h_3^0 satisfy case* (iii) *of Lemma* 5.2 *then k_3^0 and l_3^0 satisfy case* (i) (*with g_3^1 and h_3^1 interchanged*).

PROOF. (a) We suppose $g_3^0 = 2x + y$, $h_3^0 = x + 2y$, $x \neq y$ and $3x \equiv 3y \equiv g_3^1$. The following is impossible.

$$k_3^0 = 2x' + y', \quad l_3^0 = x' + 2y', \quad x' \neq y', \quad 3x' \equiv 3y' \equiv h_3^1.$$

For otherwise there is a canonical divisor $g_3^0 + k_3^0 = 2x + y + 2x' + y'$. Since $3x = g_3^1$ it follows that $x = x'$ or $x = y'$ or $3x \equiv 3x'$, a contradiction.

(b) Suppose $g_3^0 = 2x + y$, $h_3^0 = y + 2z$, $x \neq y \neq z \neq x$, $g_3^1 \equiv x + y + z$, and $h_3^1 \equiv 3x \equiv 3y \equiv 3z$. Then $k_3^0 = 2x + z$ and $l_3^0 = x + 2z$, which is case (i) of Lemma 5.2.

(c) Suppose $g_3^0 = x_1 + x_2 + x_3$, $h_3^0 = y_1 + y_2 + y_3$ where $g_3^1 \equiv x_1 + y_2 + y_3 \equiv y_1 + x_2 + x_3$, and $h_3^1 \equiv 3x_1 \equiv 3x_2 \equiv \cdots \equiv 3y_3$. Then $k_3^0 = y_1 + 2x_1$, and $l_3^0 = x_1 + 2y_1$, which is case (i) of Lemma 5.2. Q.E.D.

THEOREM 5.4. *W_4 is cyclic trigonal.*

PROOF. We suppose the vanishing properties of the table but we suppose W_4 is not cyclic trigonal. Then case (iii) of Lemma 5.2 never occurs. Consequently the pairs g_3^0, h_3^0 and k_3^0, l_3^0 in Lemma 5.3 always satisfy cases (i) and (ii) of Lemma 5.2. Thus we can suppose

$$g_3^0 = 2x + y, \quad h_3^0 = x + 2y, \quad k_3^0 = 2x + z, \quad l_3^0 = 2y + z,$$

where $3x \equiv 3y \equiv 3z \equiv h_3^1$ and $x + y + z \equiv g_3^1$.

But each of the four linear series satisfies case (i) of Lemma 5.1. For instance $g_3^0 + h_3^0 \equiv 2h_3^1$ and $g_3^0 + (2z + y) \equiv 2g_3^1$. Thus cases (ii) and (iii) of Lemma 5.1 cannot occur. This contradiction proves the theorem. Q.E.D.

If W_4 admits two sets of vanishings as in the table, both g_3^1 and h_3^1 are cyclic. They will generate a noncyclic group of order nine. The four subgroups of order three have quotients of genus 0, 0, 2, and 2, that is, the other subgroups of order three are fixed point free.

The case of a cyclic trigonal W_4 admitting one half-canonical g_3^1 remains open. The above methods do not seem to work since $2g_3^1 = g_6^3$ is not composite. These methods appear to have even less applicability for cyclic trigonal W_3's.

6. Trigonal Riemann surfaces of low genus. Let $W_{2p-1} \to W_p$ be a smooth two-sheeted covering. $J(W_{2p-1})$ is isogeneous to $J(W_p) \oplus A_{p-1}$, where A_{p-1} is a principally polarized abelian variety, known as the Prym variety, for the covering [**12**]. If W_p is a generic trigonal Riemann surface then A_{p-1} is known to be a Jacobian (of a 4-sheeted cover of $\mathbf{P}_1$) [**14**]. In general A_{p-1} is not a Jacobian. Thus if W_p is trigonal, applying the Schottky-Jung-Farkas-Rauch theorem to the special theta relations for A_{p-1} yields theta relations for W_p which we shall call *very special theta relations*. We now describe them in a little more detail.

For appropriate choices of canonical homology bases on W_{2p-1} and W_p, the theorem of Schottky-Jung-Farkas-Rauch says

$$\theta_p\begin{bmatrix} 0 & \varepsilon \\ 0 & \varepsilon' \end{bmatrix}\theta_p\begin{bmatrix} 0 & \varepsilon \\ 1 & \varepsilon' \end{bmatrix} \Big/ \theta_{p-1}^2\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$$

is a nonzero constant independent of the $(p-1)$-dimensional theta characteristic $\left[\begin{smallmatrix} \varepsilon \\ \varepsilon' \end{smallmatrix}\right]$ [**13**]. (θ_p is the theta function for $J(W_p)$ and θ_{p-1} is the theta function for A_{p-1}.) Consequently any homogeneous theta relation holding for A_{p-1} becomes a relation for $J(W_p)$ by replacing

$$\theta_{p-1}\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$$

by

$$\sqrt{\theta_p\begin{bmatrix} 0 & \varepsilon \\ 0 & \varepsilon' \end{bmatrix}\theta_p\begin{bmatrix} 0 & \varepsilon \\ 1 & \varepsilon' \end{bmatrix}}\,.$$

For genus four a special theta relation is of type $\sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3} = 0$ where each r_k is a product of eight thetanulls. Thus a very special theta relation for genus five is of the form $\sqrt[4]{S_1} + \sqrt[4]{S_2} + \sqrt[4]{S_3} = 0$ where each S_i is a product of sixteen thetanulls. For genus six the corresponding very special relation will have four terms; for genus seven, six.

Since generic Riemann surfaces of genus 4, 5, and 6 are all 4-sheeted coverings of $\mathbf{P}_1$, very special relations describe the trigonal locus in Teichmüller space for genus 5, 6, and 7, at least locally. In particular, for genus 5 one such relation will describe the trigonal locus, which has codimension one in Teichmüller space for genus 5.

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